1201
JAN 06

University College London
DEPARTMENT OF MATHEMATICS
Mid-Sessional Examinations 2006
Mathematics M12A
Friday 13 January 2006 11.30 - 1.30 or 2.05 - 4.05

All questions may be attempted but only marks obtained on the the best **four** *solutions will count.*

The use of an electronic calculator is **not** *permitted in this examination.*

1) (i) Negate the following formula, and replace it by an equivalent one which does not involve $\neg$, , $\wedge$ or $\vee$;

$$(\forall y)(\exists x)\neg\,(P(x) \wedge \neg Q(y)) \;\bigwedge\; (\forall x)(\exists y)\,(Q(y) \wedge \neg P(y))\,.$$

(ii) Let $f : A \to B$ be a mapping between sets A, B. Explain what is meant by saying that (a) f is injective ; (b) f is invertible and show that an invertible mapping is injective.

Show that the mapping $f : \mathbf{Z} \to \mathbf{Z}$; $f(x) = 3x + 1$ is injective and decide whether or not it is surjective.

Decide with proof whether the mapping g below is injective

$$g : \mathbf{R} \to \mathbf{R} \quad ; \quad g(x) = \frac{x}{x^2 + 1}.$$

2) Let $\epsilon(r, s)$ be the basic $m \times m$ matrix given by $\epsilon(r, s)_{ij} = \delta_{ri}\delta_{sj}$ where 'δ' denotes the Kronecker delta. Explain without proof how to calculate the product $\epsilon(r, s)\epsilon(u, t)$.

Describe in detail the elementary $m \times m$ matrices

(i) $E(r, s; \lambda)$ $(r \neq s)$; (ii) $\Delta(r, \lambda)$ $(\lambda \neq 0)$; (iii) $P(r, s)$ $(r \neq s)$

in terms of the basic matrices $\epsilon(r, s)$.

For the matrix A below, find A^{-1} and express A^{-1} as a product of elementary matrices; hence also express A as a product of elementary matrices.

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

PLEASE TURN OVER

3) Let V, W be vector spaces over a field $\mathbf{F}$ and let $T : V \to W$ be a mapping; explain what is meant by saying that T is *linear*.
When T is linear, explain what is meant by

(a) the kernel, $\mathrm{Ker}(T)$ and
(b) the image, $\mathrm{Im}(T)$.

State and prove a relationship which holds between $\dim \mathrm{Ker}(T)$ and $\dim \mathrm{Im}(T)$.

Let $T_A : \mathbf{Q}^5 \to \mathbf{Q}^4$ be the linear mapping $T_A(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 & 1 \\ -1 & -2 & 1 & 0 & 1 \\ 1 & 2 & -1 & 1 & 2 \\ 2 & 4 & 0 & 1 & 5 \end{pmatrix}.$$

Find (i) $\dim \mathrm{Ker}(T_A)$; (ii) a basis for $\mathrm{Ker}(T_A)$; (iii) a basis for $\mathrm{Im}(T_A)$.

4) Let $T : U \to V$ be a linear map between vector spaces U , V, and let $\mathcal{E} = (e_i)_{1 \le i \le m}$ be a basis for U and $\Phi = (\varphi_j)_{1 \le j \le n}$ be a basis for V. Explain what is meant by the matrix $m(T)_{\mathcal{E}}^{\Phi}$ of T taken with respect to $\mathcal{E}$ (on the left) and Φ (on the right) and prove that if $S : V \to W$ is also a linear map and $\Psi = (\psi_k)_{1 \le k \le p}$ is a basis for W then

$$m(S \circ T)_{\mathcal{E}}^{\Psi} = m(S)_{\Phi}^{\Psi} m(T)_{\mathcal{E}}^{\Phi}.$$

Let $\mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$; $\Phi = \left\{ \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$
be bases for $\mathbf{F}^3$ and let $T : \mathbf{F}^3 \to \mathbf{F}^3$ be the mapping

$$T \begin{pmatrix} x_1 \\ x_2 \\ 3 \end{pmatrix} = \begin{pmatrix} & 2x_2 & +x_3 \\ -4x_1 & +5x_2 & +2x_3 \\ 2x_1 & & +x_3 \end{pmatrix}.$$

Write down (i) $m(T)_{\mathcal{E}}^{\mathcal{E}}$ and (ii) $m(\mathrm{Id})_{\Phi}^{\mathcal{E}}$, and hence find $m(T)_{\Phi}^{\Phi}$.

CONTINUED

5) Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a subset of a vector space V; explain what is meant by saying that the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent.

In each case below, decide with justification whether the given vectors are linearly independent. If they are not, give an explicit dependence relation between them.

(a) $\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -2 \end{pmatrix};$

(b) $\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ -1 \\ 1 \end{pmatrix};$

Explain what is meant by a *spanning set* for a vector space V. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a spanning set for V, and suppose that $\mathbf{u} \in V$ can be expressed as a linear combination of the form

$$\mathbf{u} = \sum_{r=1}^{n} \lambda_r \mathbf{v}_r$$

with $\lambda_1 \neq 0$. Show that $\{\mathbf{u}, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is also a spanning set for V.
State and prove the Exchange Lemma.

PLEASE TURN OVER

6) (i) Let σ be a permutation of the set $\{1, \ldots, n\}$; explain what is meant by saying that (a) σ is a transposition; (b) σ is an adjacent transposition.

Show that any transposition can be written as a product of adjacent transpositions.

(ii) Decompose $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 3 & 13 & 8 & 10 & 1 & 7 & 6 & 5 & 11 & 9 & 4 & 2 & 12 \end{pmatrix}$

into a product of disjoint cycles and hence compute $\mathrm{sign}(\sigma)$ and $\mathrm{ord}(\sigma)$.

(iii) Let $\mathcal{P}_7(\mathbf{R})$ be the vector space of polynomials of degree ≤ 7 over the field $\mathbf{R}$ and let $D : \mathcal{P}_7(\mathbf{R}) \to \mathcal{P}_7(\mathbf{R})$ be the linear map given by differentiation. Write down the least positive integer n for which $D^{3n} = 0$ on $\mathcal{P}_7(\mathbf{R})$.

By factorisation of the formal expression $D^{3n} - \mathrm{I}$, or otherwise, show that the mapping

$$D^6 + D^3 + \mathrm{I} : \mathcal{P}_7(\mathbf{R}) \to \mathcal{P}_7(\mathbf{R})$$

is invertible, and write down

(i) an expression for its inverse in terms of D, and

(ii) the unique solution $\alpha \in \mathcal{P}_7(\mathbf{R})$ to the differential equation

$$\frac{d^6\alpha}{dx^6} + \frac{d^3\alpha}{dx^3} + \alpha = x^7.$$

END OF PAPER